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REMARKS ON BIFURCATION FOR ELLIPTIC OPERATORS WITH ODD NONLINEARITY

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ABSTRACT

On estimating the eigenvalues for a class of semilinear elliptic operators, we obtain bifurcation and comparison results concerning the eigenvalues of some related linear problem.

1. Introduction

Let Ω be a bounded open subset of $\mathbf{R}^N$ ($N \geqq 3$) with smooth boundary $\partial\Omega$. Consider the semilinear eigenvalue problem

$$(1) \qquad \begin{cases} Lu + f(x, u) = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $Lu = -\sum_{i,j=1}^{N} D_i(a_{i,j}(x)D_j u) + a_0(x)u$ is a formally selfadjoint operator with bounded measurable coefficients such that $a_{i,j} = a_{j,i}(i, j = 1, \ldots, N)$ and $f: \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function (i.e. measurable in x for all $u \in \mathbf{R}$ and continuous in u for a.a. $x \in \Omega$) such that $f(x, 0) = 0$, so that $u = 0$ solves trivially (1) for each μ.

If f is odd in u and satisfies suitable growth restrictions, then by Liusternik–Schnirelmann (LS) theory one can establish, given any $r > 0$, the existence of infinitely many eigenvalues $\mu_n(r)$ ($n = 1, 2, \ldots$) for (1) associated with eigenfunctions $u_n(r)$ satisfying $\int_\Omega u_n^2(r) = r^2$.

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A natural problem in this context is to compare the eigenvalues $\mu_n(r)$ with those of some linear problem close, in a sense, to (1).

Very recently, in his investigation on the limit points of the $\mu_n(r)$, Shibata [5] has shown in particular that if f is C^1 in u, if $\partial_u f(x, u)$ satisfies a "coercivity" condition for $u > 0$, and if moreover

$$|\partial_u f(x, u) - \partial_u f(x, 0)| \leqq c\, |u|^{p-1} \tag{2}$$

for some $c \geqq 0$ and some $p: 1 < p < 1 + 2/N$, then $\mu_n(r) \to \lambda_n$ as $r \to 0$, where λ_n is the nth eigenvalue of the linear problem

$$\begin{cases} Lu + \partial_u f(x, 0)u = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

More precisely, he shows (see (ii) on p. 423 of [5]) that, as $r \to 0$,

$$\mu_n(r) = \lambda_n + O(r^{p-1}). \tag{4}$$

It is our aim to generalize this result on simplifying substantially the assumptions on f, in particular as far as regularity and growth restriction are concerned. We show indeed that, under the sole assumption

$$|f(x, u) - q(x)u| \leqq c\, |u|^p \tag{5}$$

for some $q \in L^\infty(\Omega)$ and some $p: 1 < p < 1 + 4/N$, a result like (4) holds with λ_n the eigenvalues of the linear operator $Lu + q(x)u$ in Ω subject to zero Dirichlet b.c. on $\partial\Omega$. Obviously, when f is differentiable, then (2) implies (5) with $q(x) = f'_u(x, 0)$. Let us remark here that condition (2), which in [5] is assumed to hold only for $u > 0$, must then necessarily hold for any u because $\partial_u f(x, u)$ is even in u, due to the oddness assumption on f.

In our approach, the above estimate is an immediate consequence of a bifurcation result (Theorem 2 below) concerning (1), which states that if $|f(x, u)| \leqq a\, |u|^p$ for some $p: 1 < p < 1 + 4/N$, then for each n $\mu_n(r) = \mu_n^\circ + O(r^{p-1})$ and $u_n(r) \to 0$ in $\mathring{W}^{1,2}(\Omega)$ as $r \to 0$; here μ_n° denote the eigenvalues of $Lu = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, while $\mathring{W}^{1,2}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the usual Sobolev space $W^{1,2}(\Omega)$.

When comparing with [5], we see that the bifurcation aspect is not considered there; moreover, we rely on a more appropriate use of a basic inequality concerning bounds for the L^p norm of $u \in \mathring{W}^{1,2}$ $(1 \leqq p \leqq \bar{p} := 2N/(N-2))$ in terms of the L^2 norm of u and ∇u; this simplifies considerably the estimates and, hopefully, makes the argument more transparent.

This paper complements the results given by the author in [2], where a wider range of p was allowed but at the expenses of an additional sign assumption on f. However, in [2] the emphasis was on the asymptotic distribution of the $\mu_n(r)$ as $n \to \infty$ ($r > 0$ fixed): it may be of interest to note incidentally that the range of p considered here ($1 < p < 1 + 4/N$) is the same under which the "non-linear" eigenvalues $\mu_n(r)$ obey the classical asymptotic law

$$\mu_n(r) = kn^{2/N} + \text{remainder} \qquad (n \to \infty)$$

known for the eigenvalues of the linear operator L; see [2].

2. Preliminaries

We shall use repeatedly the following result, which is a direct consequence (via Hölder's inequality) of the Sobolev embedding thoerem (see e.g. [2]):

LEMMA 1. *Let $p: 1 \leqq p \leqq p_0 := (N+2)/(N-2)$ (so that $2 \leqq p+1 \leqq \bar{p}$) and let $\beta = \beta(p) = (N/\bar{p})(\bar{p} - (p+1))$. Then, for each $\gamma: 0 \leqq \gamma \leqq \beta$, there exists $c > 0$ such that*

$$\| u \|_{p+1}^{p+1} \leqq c \| \nabla u \|_2^{p+1-\gamma} \| u \|_2^{\gamma} \tag{6}$$

for all $u \in \mathring{W}^{1,2}(\Omega)$. (Here and henceforth $\| u \|_p$ denotes the norm of u in $L^p(\Omega)$.)

We remark on passing that (6) can also be derived from the following interpolation inequality, quoted and used in [5]:

$$\| u \|_{1/\mu} \leqq c \| u \|_{1/\lambda} \| u \|_{1/\nu}^{\beta} \qquad (u \in \mathring{W}^{1,2}(\Omega))$$

whenever $1/\bar{p} \leqq \lambda < \mu < \nu$, with $\alpha = (\nu - \mu)/(\nu - \lambda)$, $\beta = (\mu - \lambda)/(\nu - \lambda)$.

Let us now go back to problem (1). To prove the existence of the eigenvalues for (1) we make use of the LS critical point theory: standard references for this are e.g. [3] or [4].

The following assumptions will be made throughout:

(A1) L is uniformly elliptic in Ω: there exists $\nu > 0$ such that, for all $x \in \Omega$ and for all $\xi \in \mathbf{R}^N$,

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geqq \nu \sum_{i=1}^{N} \xi_i^2 .$$

(A2) $f: \Omega \times \mathbf{R} \to \mathbf{R}$ is odd in u ($f(x, -u) = -f(x, u)$) and satisfies

$$|f(x,u)| \leqq a|u|^p + b$$

for some $a, b \geqq 0$ and some $1 \leqq p < 1 + 4/N$.

We shall henceforth consider weak solutions of (1), namely $u \in \mathring{W}^{1,2}(\Omega)$ such that

$$\sum_{i,j=1}^{N} \int a_{i,j}(x) D_i u D_j v + \int a_0(x) uv + \int f(x,u) v = \mu \int uv \tag{7}$$

for all $v \in \mathring{W}^{1,2}(\Omega)$: $\int$ stands for $\int_\Omega$. Let us further set

$$\phi_0(u) = \frac{1}{2} \sum_{i,j=1}^{N} \int a_{i,j}(x) D_i u D_j u + \frac{1}{2} \int a_0(x) u^2 \tag{8}$$

and

$$\phi(u) = \phi_0(u) + \int F(x,u) \tag{9}$$

where $F(x,u) = \int_0^u f(x,s)ds$. For $r > 0$ let moreover

$$M_r := \left\{ u \in \mathring{W}^{1,2}(\Omega) : \int u^2 = r^2 \right\}$$

and for each $n = 1, 2, \ldots$ set

$$K_n(r) = \{K \subset M_r : K \text{ compact, symmetric, } \gamma(K) = n\}$$

where $\gamma(K)$ denotes the genus of K. Finally, introduce the "LS critical levels"

$$C_n(r) = \inf_{K_n(r)} \sup_{K} 2\phi(u). \tag{10}$$

With these notations, we can now state the basic existence result:

THEOREM 1. *Assume* (A1), (A2). *Then given $r > 0$, there exists a sequence $u_n(r)$ $(n = 1, 2, \ldots)$ of (weak) eigenfunctions of* (1) *belonging to M_r and such that*

$$2\phi(u_n(r)) = C_n(r) \tag{11}$$

where $C_n(r)$ is as in (10); *the eigenvalue $\mu_n(r)$ corresponding to $u_n(r)$ satisfies*

$$r^2 \mu_n(r) = 2\phi_0(u_n(r)) + \int f(x, u_n(r)) u_n(r). \tag{12}$$

PROOF. We first show that ϕ is bounded below on M_r (for each r). To this purpose, note that by (A1)

$$2\phi(u) \geqq \nu \int |\nabla u|^2 - d \int u^2 - 2 \int |F(x,u)|$$

where $d = \| a_0 \|_{L^\infty(\Omega)}$. Moreover, (A2) and Schwarz' inequality imply that

$$\int |F(x,u)| \leqq a \int |u|^{p+1} + b\left(\int u^2\right)^{1/2}$$

for some new constants $a, b > 0$. Next, we use the inequality (6) with $\gamma = \beta$: on setting $2\alpha = p + 1 - \beta = (p-1)N/2$, this becomes

$$\int |u|^{p+1} \leqq c \, \| \nabla u \|_2^{2\alpha} \, \| u \|_2^{\beta}$$

and we conclude that, on M_r,

$$2\phi(u) \geqq \nu \, \| \nabla u \|_2^2 - acr^{\beta} \, \| \nabla u \|_2^{2\alpha} - dr^2 - br.$$

The assumption $p < 1 + 4/N$ is equivalent to $\alpha < 1$; the claim now follows from the above inequality.

The rest of the proof (which consists in verifying the "Palais–Smale condition" and applying the LS theory on the manifold M_r) now runs along the lines already shown in [1] and will be omitted. We only remark that by Hölder's inequality, for all $1 \leqq p \leqq p_0$ and all $u, v \in \mathring{W}^{1,2}(\Omega)$,

$$\int |u|^p v \leqq \left(\int |u|^{p+1}\right)^{p/(p+1)} \left(\int |v|^{p+1}\right)^{p/(p+1)}$$

$$= \| u \|_{p+1}^{p} \, \| v \|_{p+1}.$$

Recalling the growth assumption in (A2), this permits one to introduce, as in [1], an operator B_1 by the rule

$$(B_1(u), v) = \int f(x,u)v \qquad (u, v \in \mathring{W}^{1,2}(\Omega))$$

which turns out to be compact (for $p < p_0$) by virtue of the compact imbedding of $\mathring{W}^{1,2}(\Omega)$ into $L^p(\Omega)$ $(1 \leqq p < \bar{p})$. Therefore, the restriction $p < 1 + N/(N-2)$ in Lemma 2.2 of [5] is superfluous. We remark on passing that also the condition $a_0 \geqq 0$ on the zeroth order coefficient in L has been removed.

REMARK 1. To obtain (12), just put $u = v = u_n(r)$ in (7) and use the normalization condition $\int u_n^2(r) = r^2$.

REMARK 2. If $f \equiv 0$, the LS procedure gives exactly the eigenvalues μ_n° of $Lu = \mu u$ in Ω, $u = 0$ on $\partial\Omega$: we have in this case

$$r^2 \mu_n^\circ = \inf_{K_n(r)} \sup_K 2\phi_0(u) \tag{13}$$

which is nothing but a reformulation of the classical Courant's minimax principle in terms of the sets $K_n(r)$: see e.g. [1].

3. Results

THEOREM 2. *Let the assumptions of Theorem 1 be satisfied with $p > 1$ and $b = 0$ in the growth assumption (A2). Then each μ_n° is a bifurcation point (in $\mathring{W}^{1,2}(\Omega)$) for (1); more precisely, for each $n = 1, 2, \dots$ the eigenvalue–eigenfunction pairs $(\mu_n(r), u_n(r))$ given by Theorem 1 satisfy $\mu_n(r) = \mu_n^\circ + O(r^{p-1})$ and $\| \nabla u_n(r) \|_2 \to 0$ as $r \to 0$.*

PROOF. We shall use the inequality

$$\| u \|_{p+1}^{p+1} \leqq c \| \nabla u \|_2^2 \| u \|_2^{p-1} \qquad (u \in \mathring{W}^{1,2}(\Omega)) \tag{14}$$

which is the special case of (6) when $\gamma = p - 1$; this choice is possible since $p - 1 < \beta$ iff $p < 1 + 4/N$.

Fix $n = 1, 2, \dots$ and let us first prove that, for small $r > 0$,

$$|\mu_n(r) - \mu_n^\circ| \leqq c r^{p-1} \tag{15}$$

(here and henceforth c, d will denote some, but not always the same, positive constants, possibly depending on n). To this purpose, note that

$$|\phi(u) - \phi_0(u)| = \left| \int F(x, u) \right| \leqq c \int |u|^{p+1}. \tag{16}$$

Next we use (14) and recall that, by ellipticity,

$$2\phi_0(u) \geqq \nu \| \nabla u \|_2^2 - d \| u \|_2^2$$

so that

$$\int |u|^{p+1} \leqq c(\phi_0(u) + d \| u \|_2^2) \| u \|_2^{p-1}$$

whence, for all $u \in M_r$,

$$\int |u|^{p+1} \leqq c r^{p-1} \phi_0(u) + d r^{p+1}. \tag{17}$$

Therefore from (16),

$$-dr^{p+1}+(1-cr^{p-1})\phi_0(u)\leqq\phi(u)\leqq(1+cr^{p-1})\phi_0(u)+dr^{p+1}. \tag{18}$$

Let $r>0$ be so small that $1-cr^{p-1}>0$; on taking $\inf_{K_n(r)}\sup_K$ of each term in (18) and using (10) and (13) we then get

$$-dr^{p+1}+(1-cr^{p-1})r^2\mu_n^\circ\leqq C_n(r)\leqq(1+cr^{p-1})r^2\mu_n^\circ+dr^{p+1},$$

i.e.

$$|C_n(r)-r^2\mu_n^\circ|\leqq cr^{p+1}. \tag{19}$$

On the other hand, using (11) and (12) we have

$$\begin{aligned}|C_n(r)-r^2\mu_n(r)|&=\left|2\int F(x,u_n(r))-\int f(x,u_n(r))u_n(r)\right|\\&\leqq c\int|u_n(r)|^{p+1}.\end{aligned} \tag{20}$$

The first inequality in (18) now gives, for r sufficiently small,

$$\phi_0(u)\leqq c'\phi(u)+d'r^{p+1}\qquad(u\in M_r)$$

with c', d' new constants. Therefore, from (17),

$$\int|u|^{p+1}\leqq cr^{p-1}\phi(u)+dr^{p+1}$$

(note that $r^{2p}=O(r^{p+1})$ since $p>1$). Writing this for $u=u_n(r)$ and using (20) we get

$$|C_n(r)-r^2\mu(r)|\leqq cr^{p-1}C_n(r)+dr^{p+1}.$$

However, from (19), $C_n(r)\leqq r^2\mu_n^\circ+dr^{p+1}$ and so

$$|C_n(r)-r^2\mu_n(r)|\leqq cr^{p+1}. \tag{21}$$

(15) now follows on using (19) and (21).

Let us conclude the proof of Theorem 2. We now know that, since $\int u_n^2(r)=r^2\to0$ and $\mu_n(r)\to\mu_n^\circ$ as $r\to0$, there is L^2 bifurcation from each μ_n°. Moreover, from (12) we get

$$2\phi_0(u_n(r))\leqq r^2\mu_n(r)+\int|u_n(r)|^{p+1}$$

and hence

$$\nu\int|\nabla u_n(r)|^2-dr^2\leqq r^2\mu_n(r)+cr^{p-1}\int|\nabla u_n(r)|^2,$$

i.e.,

$$(\nu - cr^{p-1}) \int |\nabla u_n(r)|^2 \leqq r^2 \mu_n(r) + dr^2.$$

This implies $\int |\nabla u_n(r)|^2 \to 0$ as $r \to 0$, whence the result.

COROLLARY. *Assume there exists $q \in L^\infty(\Omega)$ such that*

$$|f(x,u) - q(x)u| \leqq c|u|^p$$

for some $c \geqq 0$ and some $p : 1 < p < 1 + 4/N$. Let λ_n $(n = 1, 2, \ldots)$ denote the eigenvalues of

$$\begin{cases} Lu + q(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Then $\mu_n(r) \to \lambda_n$ as $r \to 0$ for each n; more precisely, $\mu_n(r) = \lambda_n + O(r^{p-1})$ as $r \to 0$.

PROOF. Let $\tilde{L} = L + q$ and write (1) as $\tilde{L}u + g(x,u) = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, where $g(x,u) := f(x,u) - q(x)u$. Then $\tilde{L}$ satisfies (A1) and g satisfies (A2); the conclusion now follows from the previous result applied to $\tilde{L}$ and g.

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